

# OPTIMAL QUANTUM STATE DETERMINATION BY CONSTRAINED ELEMENTARY MEASUREMENTS

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ABSTRACT. The purpose of this short note is to utilize work on isotropic lines in [1], on Wigner distributions for finite-state systems in [2], estimation of the state of a finite level quantum system based on Weyl operators in the  $L^2$ -space over a finite field in [3] to display maximal abelian subsets of certain unitary bases for the matrix algebra  $M_d$  of complex square matrices of order  $d > 3$ ; and then, combine these special forms with constrained elementary measurements to obtain optimal ways to determine a  $d$ -level quantum state. This enables us to generalise illustrations and strengthen results related to quantum tomography in [4].

## 1. INTRODUCTION

Ivanovic [5], Wootters and Fields [6], Bandyopadhyay, Boykin, Roychowdhury and Vatan [7], Lawrence, Brukner and Zeilinger [8], Pittinger and Rubin [9] and many other researchers constructed mutually unbiased bases (MUB's) for a  $d$ -level quantum system with  $d$ , a prime and  $d$ , a prime power; they also pointed out obstructions to such a construction for certain composite numbers  $d$ . The starting point for them was Quantum tomography, viz., to determine a quantum state using quantum measurements that correspond to pure states and to make an attempt to find such measurements. Pure states arising from a sought after complete system of MUB's work fine for this purpose. But the problem is how

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to find them for a general  $d$ . It is well-known that MUB's are related to other equally subtle problems, of equiangular lines, for instance, one can see Calderbank, Cameron, Kantor and Seidel [10], Godsil and Roy [11], Kantor [12]. So one has to look for alternative ways.

The usual way is to begin with a unitary basis (UB), i.e., a collection of unitary operators  $\mathbf{U} = \{U_x : x \in X\}$  of unitary operators  $U_x$  on a  $d$ -dimensional Hilbert space  $\mathcal{H}$  such that  $\text{tr}(U_x^* U_y) = d\delta_{xy}$  for  $x, y \in X$ , where  $A^*$  denotes the adjoint of a linear operator  $A$  on  $\mathcal{H}$ . We may consider the case when  $I_{\mathcal{H}}$ , the identity operator on  $\mathcal{H}$  is in  $\mathbf{U}$  and then consider the unitary system  $\mathbf{W} = \{U_x : x \in X, U_x \neq I_{\mathcal{H}}\}$  instead. We can look for maximal abelian subsystems of  $\mathbf{W}$ , **W-MASS**'s so as to say. Then we can take a minimal set of **W-MASS**'s, say,  $\mathcal{V}$ , that covers  $\mathbf{W}$ . Any **W-MASS**, say,  $\mathbf{V}$  has a common orthonormal basis of eigenvectors, say,  $\mathcal{E}_{\mathbf{V}}$  and the corresponding system of one-dimensional projections, say,  $\mathcal{P}_{\mathbf{V}}$ . Then  $\mathcal{P}_{\mathcal{V}} = \cup\{\mathcal{P}_{\mathbf{V}} : \mathbf{V} \in \mathcal{V}\}$  suffices in the sense that any state  $\rho$  on  $\mathcal{H}$  is determined by  $\{\text{tr}(\rho P) : P \in \mathcal{P}_{\mathcal{V}}\}$ . The trouble is that for a composite  $d$ , the size of  $\mathcal{P}_{\mathcal{V}}$  may be more than the desired one for all  $\mathcal{V}$ 's. Such systems of smallest size are aimed at in the problem enunciated above.

Parthasarathy [3] gave a method to construct  $\mathcal{P}_{\mathcal{V}}$  of size  $(d-1) \prod_{j=1}^k (d_j+1)$ , where  $d = \prod_{j=1}^k d_j$  is the prime power factorization of  $d$  with  $d_j = p_j^{s_j}$ ,  $p_j$ 's distinct primes with the help of tensor products of Weyl operators in the  $L^2$ -spaces over the finite fields  $\mathbb{F}_{d_j}$  of cardinality  $d_j$ ,  $1 \leq j \leq k$ . Chaturvedi, Mukunda and Simon [3] termed their detailed study of the problem as Wigner distributions for finite-state systems without redundant phase-point operators, related it to isotropic lines in the lattice  $\mathbf{Z}_d \times \mathbf{Z}_d$  well-studied by Albouy [1] and provided explicit methods to obtain  $\mathcal{P}_{\mathcal{V}}$  in their set-up.

Ghosh and Singh [4], amongst other things, considered general UB's  $\mathbf{U}$  constructed from latin squares and Hadamard matrices as done by Vollbrecht and Werner [13] and Werner [14]. They obtained  $\mathbf{W}$ -MASS's for a few specific cases. They indicated a method to replace  $\mathcal{P}_{\mathcal{V}}$  by smaller subsets, say  $\mathcal{P}'_{\mathcal{V}}$ , which they illustrated only for specific examples. They also gave bounds for the size of  $\mathcal{P}'_{\mathcal{V}}$ .

We realized that relevant parts of [1], [2] and [3] can be combined with the corresponding ones, particularly, Example 3.1(vii), Example 3.2 and Illustration 4.3, in [4] to give finer results for general  $d$  in a neat manner. This short note is an attempt to display that in Sections 2, 3 and 4 respectively and explain the method of optimization in the fifth section. We follow the notation and terminology in [4].

## 2. CYCLIC GROUP CASE (OF EXAMPLE 3.1(vii) [4])

This draws upon [1]. Consider the discrete phase space  $X = \mathbb{Z}_d^2$ .

2.1. An *isotropic line* is a set of  $d$  points in the lattice  $X$  such that the symplectic product  $w(\sigma, \sigma') = mn' - m'n$  of any two points  $\sigma = (m, n)$  and  $\sigma' = (m', n')$  is zero (mod  $d$ ). The *orthogonal* of a submodule  $M$  of  $X$  is denoted  $M^w$ , i.e.,  $M^w = \{\sigma \in X : \forall \sigma' \in M, w(\sigma, \sigma') = 0 \pmod{d}\}$ .

*Isotropic submodules* are those that satisfy  $M \subset M^w$ . And *Lagrangian submodules* are the maximal isotropic submodules for inclusion, which is equivalent to  $M = M^w$ .

Albouy ([1, §2]) identifies all Lagrangian submodules, first for  $d$ , a power of prime and then for the general case  $d$  with  $\prod_{i \in I} p_i^{s_i}$ , the prime factor decomposition of  $d$ . He proves that the Lagrangian submodules are the same as isotropic lines of  $X$  and determines the number of isotropic lines of  $X$  as  $\prod_{i \in I} (p_i^{s_i+1} - 1) / (p_i - 1)$ . We just note that by definition, a  $\mathbf{W}$ -MASS together with  $(0, 0)$  in the context of

Example 3.1(vii) [4] is just a Lagrangian submodule (and vice-versa) and record the consequence that follows immediately.

2.2. (i) All **W**-MASS's are of full size  $d - 1$  and their number is  $\prod_{i \in I} (p_i^{s_i+1} - 1)/(p_i - 1)$ , where  $\prod_{i \in I} p_i^{s_i}$  is the prime power factorization of  $d$ .

(ii) This agrees with observations made in [4] and explicit layout of **W**-MASS's for the special cases  $d = 4$  and  $6$ .

To facilitate a neat picture, the lattice for  $d = 4$  was drawn differently, ordering the elements of  $\mathbb{Z}_4$  as  $0, 2, 1, 3$  in Figure 2 [4] and for similar reasons of clarity for  $d = 6$ , move-together pairs like  $\{(1, 1), (5, 5)\}, \{(1, 4), (5, 2)\}, \dots$  were drawn as single points assigned on a part of a circle, so as to say.

(iii) Albouy [1] goes on to realize enumeration of isotropic lines as orbits under the action of  $SL(2, \mathbb{Z}_d)$ . One can see [1] and [2] for more details.

(iv) Albouy [1] determines the isotropic lines through a point  $x$  in terms of  $p_i$ -valuations  $v_{p_i}(x)$  of  $x$ ,  $i \in I$ . Again, for details one can see [1] and for applications [2]. We shall come back to that in our next sections.

### 3. PAULI MATRICES TECHNIQUES (OF EXAMPLE 3.2 [4])

Ampliations of Pauli matrices (which constitute the first stage unitary bases for  $d = 2$ ) and their compositions are familiar techniques in Quantum Mechanics.

3.1. Parthasarathy [3] carried the technique further to advantage to give a unitary basis  $\mathcal{F}$  for a composite  $d = \prod_{j=1}^k p_j^{s_j}$  by identifying the  $d$ -dimensional Hilbert space  $\mathcal{H}$  with  $\bigotimes_{j=1}^k \mathcal{H}_j$ , where  $\mathcal{H}_j = L^2(\mathbb{F}_{d_j})$ ,  $\mathbb{F}_{d_j}$  being the finite field of cardinality  $d_j = p_j^{s_j}$ . He used for  $1 \leq j \leq k$ , Weyl operators  $\{W(a_j, x_j), a_j \in \mathbb{F}_{d_j} \cup \{d_j\}, x_j \in \mathbb{F}_{d_j}\}$  on  $\mathcal{H}_j$ . To elaborate, for  $q = p^s$ , a prime power, fix any non-trivial character  $\chi$  of the additive group  $\mathbb{F}_q$  and put  $\langle x, y \rangle = \chi(xy)$ ,  $x, y \in \mathbb{F}_q$ .

Then  $\langle \cdot, \cdot \rangle$  is a non-degenerate symmetric bicharacter on  $\mathbb{F}_q$ . Using the counting measure on  $\mathbb{F}_q$  and writing the indicator function of  $\{x\}$  as  $|x\rangle$ ,  $\{|x\rangle : x \in \mathbb{F}_q\}$  is an orthonormal basis for  $\mathcal{K} = L^2(\mathbb{F}_q)$ . For  $a, b \in \mathbb{F}_q$ , let  $U_a$  and  $V_b$  be the unitary operators on  $\mathcal{K}$  determined by relation  $U_a|x\rangle = |a+x\rangle$  and  $V_b|x\rangle = \langle b, x \rangle |x\rangle$ ,  $x \in \mathbb{F}_q$ . He manipulates phase factors  $\alpha(a, x)$  in terms of  $\chi$  to obtain for  $a \in \mathbb{F}_q \cup \{q\} = \tilde{\mathbb{F}}_q$ , (say),  $x \in \mathbb{F}_q$ , unitary operators on  $\mathcal{K}$  given by

$$W(a, x) = \begin{cases} \alpha(a, x) U_x V_{ax}, & \text{if } a \in \mathbb{F}_q, x \in \mathbb{F}_q \\ V_x, & \text{if } a = q, x \in \mathbb{F}_q \end{cases} \quad (3.1)$$

which in addition to satisfying Weyl communication relations, have a neat property  $W(a, x)W(a, y) = W(a, x + y)$  for  $a \in \tilde{\mathbb{F}}_q$ ,  $x, y \in \mathbb{F}_q$ .

The unitary basis so constructed for  $B(\mathcal{K})$  is simply  $\{I_{\mathcal{K}}, W(a, x) : a \in \tilde{\mathbb{F}}_q, x \in \mathbb{F}_q \setminus \{0\}\}$ .

Finally, the announced unitary basis for  $\mathcal{B}(\mathcal{H})$  is

$$\mathcal{F} = \{I_{\mathcal{H}}, W^{(i_1)}(a_{i_1}, x_{i_1}) W^{(i_2)}(a_{i_2}, x_{i_2}) \dots W^{(i_r)}(a_{i_r}, x_{i_r}), a_{i_s} \in \tilde{F}_{d_{i_s}}, x_{i_s} \in \mathbb{F}_{d_{i_s}} \setminus \{0\}, \\ s = 1, 2, \dots, r, 1 \leq i_1 < i_2 < \dots < i_r \leq k, r = 1, 2, \dots, k\}.$$

Here, for  $1 \leq j \leq k$ ,  $A \in \mathcal{B}(\mathcal{H}_j)$ ,  $A^{(j)}$  is the ampliation of  $A$  to  $\mathcal{H}$ .

For further use, we may write the members in compact form:

$$J = (\mathbf{i}) = (1 \leq i_1 < i_2 < \dots < i_r \leq k), \quad \mathbf{a} = (a_{i_s})_{s=1}^r, \\ \mathbf{x} = (x_{i_s})_{s=1}^r, \quad W(\mathbf{a}, \mathbf{x}) = \prod_{s=1}^r W^{(i_s)}(a_{i_s}, x_{i_s}),$$

which is permitted because  $W^{(i_s)}(a_{i_s}, x_{i_s})$ 's commute.

3.2. We continue with relevant formulation of excerpts from [3]. For  $q$  and other entities as in (3.1) above ([3], Theorem 2.2) can be restated as: There exist orthogonal projection operators  $\{P(a, y) : a \in \tilde{\mathbb{F}}_q, y \in \mathbb{F}_q\}$  that satisfy for  $a \in \tilde{\mathbb{F}}_q$ ,  $x, z \in \mathbb{F}_q$ ,

- (i)  $W(a, x) = \sum_{y \in \mathbb{F}_q} \langle x, y \rangle P(a, y),$
- (ii)  $P(a, z) = q^{-1} \sum_{y \in \mathbb{F}_q} \overline{\langle z, y \rangle} W(a, y),$
- (iii)  $\text{Tr } P(a, x)P(b, z) = q^{-1}$  for  $a \neq b,$
- (iv)  $P(a, x)P(a, z) = 0$  for  $x$  not equal to  $z,$  and
- (v)  $\sum_{y \in \mathbb{F}_q} P(a, y) = I_{\mathcal{K}}.$

This gives rise to projections on  $\mathcal{H}$  of the type  $P(a, \mathbf{x}) = \prod_{s=1}^r P^{(i_s)}(a_{i_s}, x_{i_s})$  on the lines of 3.1 above. Its rank is  $\prod_{j \notin \{i_1, i_2, \dots, i_r\}} d_j.$

3.3. The important point is that a density  $\rho$  on  $\mathcal{H}$  can be recovered from the probabilities  $\text{Tr } \rho P(\mathbf{a}, \mathbf{x})$  and projections  $P(\mathbf{a}, \mathbf{x})$ 's in the following sense.

For  $J = (\mathbf{i}), (1 \leq i_1 < i_2 < \dots < i_r \leq k),$  let  $S_\rho(J) = \sum \text{Tr}(\rho P(\mathbf{a}, \mathbf{x}))P(\mathbf{a}, \mathbf{x})$  with  $(\mathbf{a}, \mathbf{x})$  varying in  $\prod_{s=1}^r \tilde{\mathbb{F}}_{d_{i_s}} \times \prod_{s=1}^r (\mathbb{F}_{d_{i_s}} \setminus \{0\}).$

Then ([2], Theorem 3.1) says that

$$\rho = \sum (-1)^{k-|J|} S_\rho(J),$$

where summation is over all  $J$  as specified.

3.4. Now the ranks of the projections  $P(\mathbf{a}, \mathbf{x})$  involved are  $> 1$  unless  $r = k.$  So measurements may not be easy. On the other hand, if we consider  $J = (1 < 2 < \dots < k)$  alone, the number of these projections is larger than optimal. In fact, it requires  $\prod_{j=1}^k (d_j + 1)$  elementary measurements ([3], Remark after Theorem 3.1). We shall come back to this Quantum tomography problem later in Section 5.

#### 4. WIGNER DISTRIBUTIONS FOR FINITE-STATE SYSTEMS

With  $d$  as above in §2, viz.,  $d = \prod_{i \in I} p_i^{s_i},$  Chaturvedi, Mukunda and Simon [3] take the set up of the cyclic group  $\mathbb{Z}_d$  considered as a ring expressed as the product of  $\mathbb{Z}_{p_i^{s_i}}, i \in I,$  facilitated by the Chinese Remainder Theorem instead. They study

the Quantum tomography problem by obtaining suitable Weyl operators  $D(\sigma)$  with  $\sigma \in X$  (as in §2 above) and projection operators  $P(\lambda, i)$ ,  $i = 0, 1, \dots, d-1$ ;  $\lambda$ , any isotropic line in  $X$ . Here  $(\lambda, i) = \{\sigma + (0, i) : \sigma \in \lambda\}$ .

4.1. Really intricate crystal clear formats are obtained in [2] utilizing Albouy's streamlining indicated in §2 above. At times, the description here is more transparent. For instance, the facts like the following give us more information about isotropic lines and, as a consequence **W**-MASS's in Example 3.1(vii)[4] via the manner indicated in § 2 above.

(i) Each isotropic line is a subgroup of  $\mathbb{Z}_d^2$ . So for any  $\sigma = (m, n) \in X$ , the subgroup  $\lambda_\sigma$  generated by  $\sigma$  is the move together for  $\sigma$  (in any **W**-MASS that contains  $\sigma$ ).

(ii) For  $\sigma = (m, n) \neq (0, 0)$ , let  $h = H.C.F$  of  $m, n, d$ , counting 0 as  $d$ .

(a) If  $h = 1$ , then  $\lambda_\sigma$  is the unique isotropic line containing  $\sigma$ . All such isotropic lines are given by  $\lambda_\sigma$  with  $\sigma = (m, 1)$ ,  $m \in \mathbb{Z}_d$  and  $(1, n)$ ,  $n$ , zero or a factor of  $d$  other than  $d$ . Alternatively we may consider  $\sigma = (m, 1)$ ,  $m$  zero or a factor of  $d$  other than  $d$  and  $(1, n)$ ,  $n \in \mathbb{Z}_d$ .

(b) If  $h > 1$ , then  $|\lambda_\sigma| = d/h$  and  $\lambda_\sigma$  is a move together of  $\sigma$  in different isotropic lines that contain  $\sigma$ . This gives different **W**-MASS's overlapping, of course, in  $\{U(m, n), (m, n) \in \lambda_\sigma\}$ .

4.2. In case of overlapping  $\lambda$ 's and accordingly overlapping **W**-MASS's, the condition of unbiasedness assumes a modified form. The expression (109) of [2] can be written as:

$$\text{Tr}(P_{(\lambda, i)} P_{(\lambda', j)}) = \frac{1}{d} [\text{number of points common to the lines } (\lambda, i) \text{ and } (\lambda', j)].$$

4.3. A rough estimate for the number of probabilities from the said projection operators indexed by  $\{(\lambda, i) : \lambda \text{ any isotropic line, } i = 0, 1, \dots, d-1\}$  is more than optimal, of course, except when  $d$  is a prime. This brings us to the technique displayed in §4.3 [4] to reduce the job.

## 5. CONSTRAINED ELEMENTARY MEASUREMENTS AND QUANTUM TOMOGRAPHY

We begin with a definition.

**Definition 5.1** (Constrained elementary measurement).

Let  $\mathbf{Q} = \{Q_t : 1 \leq t \leq \tau\}$  be a family of mutually orthogonal projections on a Hilbert space  $\mathcal{K}$  of dimension  $q$  that add upto  $I_{\mathcal{K}}$ . A family  $\mathcal{P} = \{P_j : 0 \leq j \leq q-1\}$  of mutually orthogonal rank one projection's adding upto  $I_{\mathcal{K}}$  will be said to be **Q-constrained** if  $\{j : 0 \leq j \leq q-1\}$  can be decomposed as (a disjoint union of)  $\{I_t : 1 \leq t \leq \tau\}$  with  $\sum_{j \in I_t} P_j = Q_t, 1 \leq t \leq \tau$ .  $\mathcal{P}$  will be called a **Q-constrained elementary measurement**.

5.1. Let  $\{\mathcal{P}_v : 1 \leq v \leq g\}$  be **Q-constrained elementary measurements** with  $\mathcal{P}_v = \{P_j^v : 0 \leq j \leq q-1\}$  and decomposition  $\{I_t^v : 1 \leq t \leq \tau\}, 1 \leq v \leq g$ . Suppose  $g \geq 2, \tau \geq 2$  and  $T = \{t : \dim Q_t > 1\} \neq \emptyset$ . Let  $T' = \{t : 1 \leq t \leq \tau, t \notin T\}$ . For  $2 \leq v \leq g, t \in T$ , fix any  $j_t^v \in I_t^v$  and set  $J_t^v = I_t^v \setminus \{j_t^v\}$ . For  $t \in T'$ , let  $J_t^v = I_t^v$ . Then the family  $\{\mathcal{P}_v : 1 \leq v \leq g\}$  can be replaced by the smaller family  $\mathcal{P}' = \mathcal{P}_1 \cup \bigcup_{2 \leq v \leq g} \bigcup_{t=1}^{\tau} \{P_j^v : j \in J_t^v\}$  for estimation purposes.

To see this, we only have to note that missing  $P_{j_t^v}^v$  is  $Q_t - \sum_{j \in J_t^v} P_j^v = \sum_{j \in I_t^1} P_j^1 - \sum_{j \in J_t^v} P_j^v$ .



5.2. As already noted in ([4], Theorem 2.8) and utilized in 4.3(iii) [4] unitary operators in a unitary basis that are common to different **W**-MASS's have multiple eigenvalues. The projection on the corresponding eigenspace can be taken to be  $Q_t$ . As explained in [1] and [2] (and already noted in § 2 and § 4 above) for the case of Example 3.1(vii) the overlapping members also constitute a subgroup of  $X = \mathbb{Z}_d^2$ . If this subgroup is cyclic, we may take a generator  $U$  for this purpose and decompose  $\mathcal{H}$  into eigenspaces of  $U$  and thus have  $\mathbf{Q} = \{Q_t : 1 \leq t \leq \tau\}$  with  $Q_t$ 's projections on the eigenspaces. We may now take  $\mathcal{P}_v$  arising from a common complete system of unit eigenvectors for a **W**-MASS  $V_v$  which contains  $U$ ,  $1 \leq v \leq g$ . We are in the situation of 5.1 above. So the club  $\mathcal{P} = \{\mathcal{P}_v : 1 \leq v \leq g\}$  of elementary measurements can be replaced by  $\mathcal{P}'$  as in 5.1 above. Hence the total size of the set of rank one projections needed can be reduced.

5.3. As seen in §4.3 [4] for  $d = 4$  the technique explained above in 5.2 gives the optimal size itself, whereas for  $d = 6$ , we had a larger number still.

More careful study can perhaps reduce the size to optimal.

5.4. More elaborate study can reduce the size by taking isotropic lines with overlaps as explained in [2] indicated in §4 above.

5.5. We now come to §3 situation.

$$W(\mathbf{a}, \mathbf{x})W(\mathbf{b}, \mathbf{y}) = W(\mathbf{b}, \mathbf{y})W(\mathbf{a}, \mathbf{x})$$

if and only if for common indices  $i$  for  $(\mathbf{a}, \mathbf{x})$  and  $(\mathbf{b}, \mathbf{y})$ .  $W(a_i, x_i)W(b_i, y_i)$  is a multiple (say,  $\lambda_i \in S^1$ , the unit circle) of  $W(b_i, y_i)W(a_i, x_i)$  and  $\prod \lambda_i$  of such multiples is 1. So the first requirement that works fine in view of 2.2 is that for common  $i$ 's,  $a_i = b_i$ .

(i) We may partition  $\{1, 2, \dots, k\}$  into sets  $\{I_1, \dots, I_\tau\}$  and thus have commuting  $W(\mathbf{a}, \mathbf{x})$ 's confining attention to different  $I_j$ 's at a time.

(ii)  $W(\mathbf{a}, \mathbf{x})$  has eigenvalue  $\prod_{j=1}^r \langle x_{i_j}, y_{i_j} \rangle$  with  $\prod_{j=1}^r P(a_{i_j}, y_{i_j})$  as eigen projection (or subeigenprojection !). For  $r < k$ , this does give us a multiple eigenvalue for  $W(\mathbf{a}, \mathbf{x})$  which helps us to reduce the size by considering **W**-MASS's containing  $W(\mathbf{a}, \mathbf{x})$  in a minimal set of **W**-MASS's whose union is **W**.

5.6. We can use the crude estimates  $\delta$  like  $\prod_{j=1}^k (d_j + 1)$  as in [3] and  $\prod_{j=1}^k (p_i^{s_i+1} - 1)/(p_i - 1)$  as in [2] (see §3 and §4 above) and different ones given in ([1], [2]) for the size of a minimal set of **W**-MASS's whose union is **W**.

4.3(iv) [4] then gives the following.

For  $d = 2r$  with  $r$  odd  $\geq 3$ , there exists a pure POVM of size  $\leq 4 + (d - 2)\delta$ . This estimate using [2] is significantly smaller than that in [4] simply because this is quadratic whereas the one in [4] is cubic in nature.

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## REFERENCES

- [1] O. Albouy, The isotropic lines of  $\mathbb{Z}_d^2$ , J. Phys. A. Math. Theor., 42 (2009), 072001.

- [2] S. Chaturvedi, N. Mukunda and R. Simon, Wigner distributions for finite-state systems without redundant phase-point operators, *J. Phys. A: Math. Theor.* 43 (2010), 0753075302 (21 pp).
- [3] K.R. Parthasarathy, On estimating the state of a finite level quantum system, *Inf. Dimens. Anal. Quantum Prob. Relat. Top.*, Vol. 07, Issue 4, Dec. 2004. 607-617.
- [4] Sibasish Ghosh and Ajit Iqbal Singh, Invariants for maximally entangled vectors and unitary bases, arXiv:1401.0099v1 [quant-ph] 31 Dec 2013.
- [5] I.D. Ivanovic, Geometrical description of quantal state determination, *J. Phys. A: Math. Gen.* 14 (1981), 3241-3245.
- [6] W.K. Wootters and B.D. Fields, Optimal state determination by mutually unbiased measurement, *Ann. Phys.* 1, 91 (1989), 363-381.
- [7] S. Bandyopadhyay, P.O. Boykin, V. Roychowdhury and Farrokh Vatan, A new proof of mutually unbiased bases, *Algorithmica*, 34 (2002), 512-530. arXiv: quant-ph/0103162 v3, 7 September 2001.
- [8] J. Lawrence, C. Brukner and A. Zeilinger: Mutually unbiased binary observable sets on  $N$  qubits, *Phys. Rev. A* 65 (2002), 032320, arXiv:quant-ph/0104012 v2, 30 April 2002.
- [9] A.O. Pittinger and M.H. Rubin, Mutually unbiased bases, generalized spin matrices and separability, *Lin. Alg. Appl.* 390(2004), 255-278.
- [10] A.R. Calderbank, P.J. Cameron, W.M. Kantor and J.J. Seidel,  $\mathbf{Z}_4$ -Kerdock codes, orthogonal spreads, and extremal Euclidean line-sets, *Proc. LMS* 75 (1997), 436-480.
- [11] C. Godsil and A. Roy, Equiangular lines, mutually unbiased bases, and spin models, *Eur. J. Comb.* 30 (2009), 246-262.
- [12] W.M. Kantor, MUBs inequivalence and affine planes, *J. Math. Phys.* 53 (2012), 032204.
- [13] K.G.H. Vollbrecht and R.F. Werner, Why two qubits are special, *J. Math. Phys.*, Vol. 41, No. 10 (October, 2000), 6772-6782, arXiv:quant-ph/9910064.

- [14] R.F. Werner, All teleportation and dense coding schemes, J. Phys. A; MATH. Gen. 34 (2001), 7081-7094, arXiv:quant-ph/0003070.

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